(1) Q No.- Obtain Maclaurin's Series expansion of $\sin x$ over R with justification.

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or,

Prove that
$$\sin x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \cdots$$
from Maclaurin's series.

or, Discuss the Possibility of expanding $\sin x$ as a series in ascending power of $x$.

Soln:- Let $f(x) = \sin x$ for all $x \in R$

$\therefore f^n(x) = \sin\left(\frac{n\pi}{2} + x\right)$ for all $x \in R$.

Thus for each +ve integer $n$, $f^n$ is defined in every interval $[-h, h]$.

Now, writing Lagrange's remainder after $n$th term.

$$R_n(x) = \frac{x^n}{\lfloor n} f^n(\theta x) \text{ when } 0<\theta<1.$$

$$= \frac{x^n}{\lfloor n} \sin\left(\frac{n\pi}{2} + \theta x\right)$$

$\because |R_n(x)| \leq \left|\frac{x^n}{\lfloor n}\right|$ for all $x \in R$.

$\therefore \underset{n\to\infty}{Lt} \frac{x^n}{\lfloor n} = 0 \quad \therefore \underset{n\to\infty}{Lt} R_n(x) = 0$.

So from Maclaurin's Series

$\therefore f(x) = f(0) + x f'(0) + \cdots + \frac{x^{n-1}}{\lfloor n-1} f^{n-1}(0) + \cdots$

--- for all $x \in R$.

Putting $f(x) = \sin x$

$\therefore f^n(n) = \sin \frac{n\pi}{2}$

$\therefore \sin x = x \times 1 - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} + \cdots \quad f'(0) = \cos 0$

for all $x \in R$

(2) Obtain Maclaurin's Series expansion of $e^x$ over $R$ with justification :

Soln- Let $f(x) = e^x$

$\therefore f^n(x) = e^x$ for each +ve integer.

Thus $f$ possesses derivationes of every order for every real $x$,

Now, Lagranges form of remainder for each fixed $x \in R$.

$R_n(x) = \frac{x^n}{\lfloor n} f^n(\theta x)$

$= \frac{x^n}{\lfloor n} . e^{\theta x}$ when, $0 < \theta < 1$.

$\therefore \underset{n\to\infty}{Lt} R_n(x) = \underset{n\to\infty}{Lt} \frac{x^n}{\lfloor n} e^{\theta x}$

$\left(\underset{n\to\infty}{Lt} \frac{x^n}{\lfloor n}\right) e^{\theta x} = 0$.

Hence all Condition of Maclaurin's infinite expn. are satisfied.

$\therefore f(0) = e^0, f^n(0) = e^0 = 1$.

for each +ve integer $n$.

Now, $f(x) = f(0) + x f'(0) + \frac{x^2}{\lfloor 2} f''(0) +$

$$= 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \cdots$$

for each $x \in R$.

Prove that,

$$\sin ax = ax - \frac{a^3x^3}{\lfloor 3} + \frac{a^5x^5}{\lfloor 5} + \cdots + \frac{a^{n-1}}{\lfloor n-1} \sin\left(\frac{n-1}{2}\pi\right) + \frac{a^n x^n}{\lfloor n} \sin\left(a\theta x + \frac{n\pi}{2}\right).$$

Soln - $f(x) = \sin ax$

$f'(x) = a\cos ax = a\sin\left(\frac{\pi}{2} + ax\right)$

$f''(x) = a^2 \sin\left(\frac{2\pi}{2} + ax\right)$

$f'''(x) = a^3 \sin\left(\frac{3\pi}{2} + ax\right)$

$f^{iv}(x) = a^4 \sin\left(\frac{4\pi}{2} + ax\right)$

$f^{n-1}(x) = a^{n-1} \sin\left(\frac{n-1}{2}\pi + ax\right)$

& $f^n(x) = a^n \sin\left(\frac{n}{2}\pi + ax\right)$

$\therefore f^n(x) = a^n \sin\left(\frac{n}{2}\pi + ax\right)$

Now, $f(0) = 0$

$f'(0) = a$

$f''(0) = 0$

$f'''(0) = -a^3$

$f^{iv}(0) = 0$

$f^{n-1}(0) = a^{n-1} \sin\left(\frac{(n-1)\pi}{2}\right)$.

Now, We know from Maclaurin's series, we have

$$f(x) = f(0) + xf'(0) + \frac{x^2}{\lfloor 2} f''(0) + \frac{x^3}{\lfloor 3} f'''(0) + \cdots + \frac{x^{n-1}}{\lfloor n-1} f^{n-1}(0) + \frac{x^n}{\lfloor n} f^n(\theta x)$$

Where $0 < \theta < 1$.

$$\therefore \sin(ax) = 0 + xa + 0 + \frac{x^3}{\lfloor 3} \times -a^3 + 0 + \frac{x^5}{\lfloor 5} a^5 + \cdots$$

$$\cdots + \frac{x^{n-1}}{\lfloor n-1} \cdot a^{n-1} \sin\frac{n-1}{2}\pi + \frac{x^n}{\lfloor n} a^n \sin\left(a\theta x + \frac{n\pi}{2}\right)$$

$$\text{or, } \sin ax = ax - \frac{a^3x^3}{\lfloor 3} + \frac{a^5x^5}{\lfloor 5} + \cdots + \frac{x^{n-1}}{\lfloor n-1} a^{n-1} \sin\left(\frac{n-1}{2}\pi\right) + \frac{x^n}{\lfloor n} a^n \sin\left(a\theta x + \frac{n\pi}{2}\right)$$

Proved

Q.N. $f(x+h) = f(x) + hf'(x) + \frac{h^2}{\lfloor 2} f''(x+\theta h)$ find the value of $\theta$ as $x \to a$, $f(x)$ being $(x-a)^{5/2}$.

Solu$^n$. It is given that,

$$f(x) = (x-a)^{5/2}$$

$$\therefore f(x+h) = (x+h-a)^{5/2}$$

$$f'(x) = \frac{5}{2}(x-a)^{3/2}$$

$$f(x+\theta h) = (x+\theta h-a)^{5/2}$$

$$f'(x+\theta h) = \frac{5}{2}(x+\theta h-a)^{3/2}$$

$$f''(x+\theta h) = \frac{15}{4}(x+\theta h-a)^{1/2}$$

$$\because f(x+h) = f(x) + hf'(x) + \frac{h^2}{\lfloor 2} f''(x+\theta h)$$

Now, substituting values from above, we have

$$(x+h-a)^{5/2} = (x-a)^{5/2} + h\frac{5}{2}(x-a)^{3/2} + \frac{h^2}{\lfloor 2} \times \frac{15}{4} (x+\theta h-a)^{1/2} \quad \text{—— (1)}$$

when $x \to a$

$$h^{5/2} = \frac{h^2}{\lfloor 2} \times \frac{15}{4} (\theta h)^{1/2}$$

$$\text{or, } h^{5/2} = \frac{h^2}{1\times 2} \times \frac{15}{4} \times \theta^{1/2} \cdot h^{1/2}$$

$$\text{or, } \cancel{h^{5/2}} = \cancel{h^{5/2}} \times \frac{15}{8} \theta^{1/2}$$

$$\text{or, } 1 = \frac{15}{8}\sqrt{\theta}$$

$$\frac{8}{15} = \sqrt{\theta}$$

on squaring, we have

$$\theta = \frac{64}{225}$$